

THE DIMINISHED BASE LOCUS IS NOT ALWAYS CLOSED

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ABSTRACT. We exhibit a pseudoeffective \mathbb{R} -divisor D_λ on the blow-up of \mathbb{P}^3 at nine very general points which lies in the closed movable cone and has negative intersections with a set of curves whose union is Zariski dense. It follows that the diminished base locus $\mathbf{B}_-(D_\lambda) = \bigcup_{A \text{ ample}} \mathbf{B}(D_\lambda + A)$ is not closed and that D_λ does not admit a Zariski decomposition in even a very weak sense. By a similar method, we construct an \mathbb{R} -divisor on the family of blow-ups of \mathbb{P}^2 at ten distinct points, which is nef on a very general fiber but fails to be nef over countably many prime divisors in the base.

1. INTRODUCTION

For a pseudoeffective \mathbb{R} -divisor D on a normal variety Y , the diminished base locus (also called the non-nef locus or restricted base locus) is the union

$$\mathbf{B}_-(D) = \bigcup_{\substack{A \text{ ample} \\ D+A \text{ } \mathbb{Q}\text{-Cartier}}} \mathbf{B}(D+A),$$

where $\mathbf{B}(D+A) = \bigcap_{n \geq 1} \text{Bs}(n(D+A))$ is the stable base locus [5]. This is at most a countable union of subvarieties, but in many examples the union is finite, i.e. Zariski closed. We will give an example of an \mathbb{R} -divisor for which this locus is not Zariski closed.

Theorem 1.1. *Let X be the blow-up of \mathbb{P}^3 at nine very general points. There exists a pseudoeffective \mathbb{R} -divisor D_λ on X with the following properties:*

- (1) *There is a countable set of curves $C_n \subset X$ with $D_\lambda \cdot C_n < 0$, whose union is Zariski dense on X .*
- (2) *$\mathbf{B}_-(D_\lambda)$ is a countable union of curves.*
- (3) *There is no decomposition $f^*D_\lambda \equiv_{\text{num}} P + N$ of f^*D_λ into nef and effective components on any birational model $f : Y \rightarrow X$.*

Further, there exists a big \mathbb{R} -divisor D'_λ on $\mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(1))$ for which $\mathbf{B}_-(D'_\lambda)$ is a countable union of curves, where $\mathcal{O}_X(1)$ is any very ample line bundle on X .

Recall the following basic property of nef divisors.

Proposition ([9], Proposition 1.4.14). Suppose that X and S are varieties and $\pi : X \rightarrow S$ is a surjective and proper morphism. Let D be a Cartier divisor on X . If D_0 is nef for some $0 \in S$, then D_s is nef for very general $s \in S$ (i.e. for all s not contained in some countable union of subvarieties).

There do not seem to be any examples known where nefness fails over a countable union of subvarieties, rather than simply a Zariski closed set. The proof in [9] remains valid if D is an \mathbb{R} -Cartier divisor on X , and the next example demonstrates that, at least in this greater generality, the “very general” of the conclusion is indeed essential.

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Theorem 1.2. *Let $\Sigma = (\mathbb{P}^2)^{10} \setminus \Delta$, where Δ is the locus where two points coincide, and let $\mathcal{X} \rightarrow \Sigma$ be the family whose fiber over $\mathbf{p} \in \Sigma$ is the blow-up of \mathbb{P}^2 at the corresponding ten points. There exists an \mathbb{R} -divisor C_λ on \mathcal{X} such that $C_{\lambda, \mathbf{p}}$ is nef for very general \mathbf{p} , but there are countably many prime divisors $V_n \subset \Sigma$ such that $C_{\lambda, \mathbf{p}}$ is not nef if $\mathbf{p} \in V_n$.*

The strategy in each case is similar to that employed by Nagata in constructing (-1) -curves of arbitrarily large degree on the blow-up of \mathbb{P}^2 at ten or more points: taking strict transforms of divisors on \mathbb{P}^n under sequences of Cremona transformations centered at various points gives rise to effective divisors with large multiplicities at prescribed points [11]. The strict transforms of these divisors on the blow-up at those points, appropriately rescaled, converge in $N^1(X)$.

Generally, suppose that $\phi : X \dashrightarrow Y$ is a rational map which is an isomorphism in codimension 1. Taking strict transforms of divisors induces a map $\phi_* : N^1(X) \rightarrow N^1(Y)$. If $X = Y$, or X and Y are both blow-ups of a fixed variety at sets of very general points, there is an identification $\Phi^{-1} : N^1(Y) \rightarrow N^1(X)$ which preserves various cones of divisors on X and Y . When the composition $\Phi^{-1} \circ \phi_* : N^1(X) \rightarrow N^1(X)$ has a unique eigenvalue with magnitude greater than one, the dominant eigenvector spans an extremal ray on the pseudoeffective cone (see Lemma 2.1). Both of the examples above arise as such eigenvectors.

I have learned that recent work of T. Bayraktar also considers the diminished base loci of a general class of \mathbb{R} -divisors constructed by the same method; the example presented here is essentially one in which the inclusion in Theorem 1.1 of [1] is an equality.

The next section contains some preliminary lemmas needed for the constructions. Section 3 provides the example of Theorem 1.2. Section 4 introduces the standard Cremona transformation $\text{Cr} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, leading to the construction of D_λ . The various claims of Theorem 1.1 are proved in Sections 5 and 6 as Lemmas 5.2, 5.4, 6.6, and 6.7.

2. PRELIMINARIES

We first record a simple observation which implies that \mathbb{R} -divisors arising as eigenvectors of automorphisms of $N^1(X)$ often generate extremal rays on the various cones of divisors.

Lemma 2.1 (cf. [3]). *Suppose that V is a finite dimensional real vector space, $G \subset V$ is a closed convex cone with nonempty interior and containing no line, and $T : V \rightarrow V$ is a linear map with $T(G) = G$. If T has a real eigenvalue λ of algebraic multiplicity one, with magnitude larger than that of any other eigenvalue, then the λ -eigenvector v_λ (with appropriate sign) spans an extremal ray on G .*

Proof. Fix a norm $|\cdot|$ on V and write $V = \mathbb{R}v_\lambda \oplus W$, where W is the direct sum of the other eigenspaces, so that $T|_W$ has norm strictly less than λ . Since G has nonempty interior, there exists $v \in G$ with nonzero component in the v_λ -eigenspace. Then $\frac{1}{\lambda^n} T^n v$ converges to some nonzero multiple of v_λ . Switching the sign if needed, we conclude that v_λ is contained in G .

Suppose that v_λ is not extremal, i.e. that there exists a nonzero $w \in W$ for which $v_\lambda + w$ and $v_\lambda - w$ are both in G . Since its image contains an open set, T is invertible and $T^{-1}(G) = G$. There is a sequence n_i for which $T^{-n_i} w / |T^{-n_i} w|$ converges to a nonzero limit $r \in V$. Since $T|_W$ has norm less than λ , $|\lambda^n T^{-n} w|$ grows without bound as n increases, and $v_\lambda / |\lambda^n T^{-n} w|$ converges to 0. It follows that the two sequences of vectors in G

$$\frac{\lambda^{n_i} T^{-n_i} (v_\lambda \pm w)}{|\lambda^{n_i} T^{-n_i} w|} = \frac{v_\lambda}{|\lambda^{n_i} T^{-n_i} w|} \pm \frac{\lambda^{n_i} T^{-n_i} w}{|\lambda^{n_i} T^{-n_i} w|}$$

converge to $\pm r$. The closedness of G implies that both r and $-r$ are contained in G , contradicting the assumption that G contains no line. \square

Both examples deal with blow-ups of projective space, and it will be useful to establish some basic notation for k -tuples of points on \mathbb{P}^n . Throughout, we work over an uncountable algebraically closed field of arbitrary characteristic. Let $\Sigma = (\mathbb{P}^n)^k \setminus \Delta$ be the set of k -tuples with all points distinct, $\pi_i : \Sigma \rightarrow \mathbb{P}^n$ the i^{th} projection, and \mathcal{X} the blow-up of $\Sigma \times \mathbb{P}^n$ along the images of the sections $(\text{id}, \pi_i) : \Sigma \rightarrow \Sigma \times \mathbb{P}^n$ for $1 \leq i \leq k$. The fiber $X_{\mathbf{p}}$ of $\pi : \mathcal{X} \rightarrow \Sigma$ over $\mathbf{p} = (p_1, \dots, p_k) \in \Sigma$ is the blow-up of \mathbb{P}^n at the corresponding k points.

If Y is a normal projective variety, the group of \mathbb{R} -divisors on Y modulo numerical equivalence is denoted $N^1(Y) = N^1(Y)_{\mathbb{R}}$, and $[D] \in N^1(Y)$ is the numerical class of a divisor D , though when no confusion is possible we omit the brackets. Dually, $N_1(Y)$ is the group of curves modulo numerical equivalence, and the class of C is written $[C]$.

For any fiber $X = X_{\mathbf{p}}$, there are decompositions $N^1(X) = \mathbb{R}H \oplus \bigoplus_{i=1}^k \mathbb{R}E_i$ and $N_1(X) = \mathbb{R}h \oplus \bigoplus_{i=1}^k \mathbb{R}e_i$, where H is the pullback of the hyperplane class on \mathbb{P}^n , h is the class of the strict transform of a line disjoint from the points of \mathbf{p} , E_i are the exceptional divisors, and e_i the classes of lines in the E_i . The intersection pairing on the exceptional classes is given by $E_i \cdot e_i = -1$. If \mathbf{p} and \mathbf{q} are two different sets of points, there is an isomorphism $\Phi_{\mathbf{p}\mathbf{q}} : N^1(X_{\mathbf{p}}) \rightarrow N^1(X_{\mathbf{q}})$ which sends $H_{\mathbf{p}}$ to $H_{\mathbf{q}}$ and $E_{i,\mathbf{p}}$ to $E_{i,\mathbf{q}}$; the matrix for $\Phi_{\mathbf{p}\mathbf{q}}$ with respect to the above bases is the $(k+1) \times (k+1)$ identity matrix.

The pseudoeffective cone $\overline{\text{Eff}}(X) \subset N^1(X)$ is the closure of the cone $\text{Eff}(X)$ generated by classes of effective Cartier divisors, and the movable cone $\overline{\text{Mov}}(X) \subset N^1(X)$ is the closure of the cone generated by classes of such divisors whose base locus has codimension at least 2. The next lemma shows that if \mathbf{p} and \mathbf{q} are very general, the movable and pseudoeffective cones of $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ coincide under the identification $\Phi_{\mathbf{p}\mathbf{q}}$.

Lemma 2.2. *There is a set $U \subset \Sigma$, the complement of a countable union of subvarieties, such that if \mathbf{p} and \mathbf{q} lie in U , then $\Phi_{\mathbf{p}\mathbf{q}}(\text{Eff}(X_{\mathbf{p}})) = \text{Eff}(X_{\mathbf{q}})$, $\Phi_{\mathbf{p}\mathbf{q}}(\overline{\text{Eff}}(X_{\mathbf{p}})) = \overline{\text{Eff}}(X_{\mathbf{q}})$, and $\Phi_{\mathbf{p}\mathbf{q}}(\overline{\text{Mov}}(X_{\mathbf{p}})) = \overline{\text{Mov}}(X_{\mathbf{q}})$.*

Proof. An integral class $D = dH - \sum a_i E_i$ has $h^0(X_{\mathbf{p}}, D_{\mathbf{p}}) > 0$ for all \mathbf{p} in some closed subset of Σ by the semicontinuity theorem. It follows that \mathbf{p} and \mathbf{q} have the same effective integral classes as long as these two points lie off of the countably many proper closed subsets that arise in this way, and so the effective and pseudoeffective cones coincide.

For the movable cone, we again restrict our attention to integral classes. An integral class $D_{\mathbf{p}}$ has base locus of codimension 2 if $h^0(X_{\mathbf{p}}, D_{\mathbf{p}}) > 1$ and $D_{\mathbf{p}}$ has no fixed part, i.e. there does not exist a nonzero effective class $F_{\mathbf{p}}$ such that $h^0(X_{\mathbf{p}}, D_{\mathbf{p}} - F_{\mathbf{p}}) = h^0(X_{\mathbf{p}}, D_{\mathbf{p}})$. The result follows as above by the fact that for any integral D and F , each of $h^0(X_{\mathbf{p}}, D_{\mathbf{p}} - F_{\mathbf{p}})$ and $h^0(X_{\mathbf{p}}, D_{\mathbf{p}})$ is constant for \mathbf{p} in some open set. \square

Let $\rho : \text{Hilb}(\mathcal{X}/\Sigma) \rightarrow \Sigma$ denote the relative Hilbert scheme parametrizing subschemes which lie in fibers of π . The union $Z \subset \Sigma$ of the images of all components of $\text{Hilb}(\mathcal{X}/\Sigma)$ which do not dominate Σ is a countable union of proper closed subsets.

Lemma 2.3. *Suppose that $\pi : Y \rightarrow S$ is a projective morphism of varieties, and $V_i \subset Y$ ($i \in I$) is a set of irreducible closed subvarieties of Y , each dominating S . Then there exists closed $V \subset Y$ and a set $U \subset S$, the complement of a countable union of proper closed subsets, such that if $s \in U$, the Zariski closure $\overline{\bigcup_i V_{i,s}} \subset Y_s$ is given by V_s .*

Proof. For each i , consider the closed set $\Phi_i = \{q \in \text{Hilb}(Y/S) : V_{i,\rho(q)} \subset H_q\}$, where $H_q \subset Y_{\rho(q)}$ is the subscheme determined by q . This is the set of families which contain V_i over points of the base where defined. Form $\Phi = \bigcap_i \Phi_i \subset \text{Hilb}(Y/S)$, which is also closed and so has countably many components. Let Z' be the union of the images in S of all components of Φ which do not dominate S ; thus if $s \in U = S \setminus Z'$, any $W_s \subset Y_s$ which contains all of the varieties $V_{i,s}$ belongs to a dominant flat family $W \rightarrow S$ such that $W_{s'}$ contains all $V_{i,s'}$ for any s' .

Let η be the generic point of S , and let V_η be the Zariski closure of $\bigcup_i V_{i,\eta}$ inside Y_η . The closure of V_η in Y is a subvariety V , dominating S . Since V_i is irreducible, we have $V_{i,s} \subset V_s$ for every $s \in S$ and i . Fix some $s \in S \setminus Z'$, and let $W_s \subset V_s$ be the Zariski closure of the $V_{i,s}$. Since s lies off of Z' , W_s belongs to a family $W \rightarrow S$ with $V_{i,\eta} \subset W_\eta$. It must be that $W_\eta \subset V_\eta$, since W_s and V_s are obtained by taking the fibers of the Zariski closures of W_η and V_η in Y , and so $W_s = V_s$ as required. \square

Example 2.4. It is possible that the dimension of the Zariski closure in the fibers jumps down at countably many points of the base, as illustrated by the following example. Let $Y = S \times \mathbb{P}^1$ for a smooth curve S , and take $\pi : Y \rightarrow S$ the projection. Fix a sequence of distinct points $x_0, x_1, \dots \in \mathbb{P}^1$, as well as a sequence of distinct points $s_0, s_1, \dots \in S$. For $n \geq 1$, choose sections $V_n : S \rightarrow Y$ such that $V_n(s_0) = x_n$, while for $1 \leq i \leq n$, we have $V_n(s_i) = x_0$. Then $\bigcup_n V_n(s_0) = \{x_n\}_{n \geq 1}$ is Zariski dense in the fiber $\pi^{-1}(s_0)$, while for $i \geq 1$, $\bigcup_n V_n(s_i) = \bigcup_{n=0}^{i-1} V_n(s_i) \cup \{x_0\}$ is a finite set.

We will use the following properties of the diminished base locus, which follow from the definition (see [5] for details).

Lemma 2.5. *Suppose that D is a pseudoeffective \mathbb{R} -divisor on a normal variety Y .*

- (1) $\mathbf{B}_-(D)$ depends only on the numerical class $[D] \in N^1(Y)$.
- (2) $\mathbf{B}_-(D) = \emptyset$ if and only if D is nef.
- (3) If C is a curve with $D \cdot C < 0$, then $C \subset \mathbf{B}_-(D)$.
- (4) $\mathbf{B}_-(D + D') \subseteq \mathbf{B}_-(D) \cup \mathbf{B}_-(D')$.
- (5) If $f : Y' \rightarrow Y$ is a surjective morphism between smooth varieties, $\mathbf{B}_-(f^*D) = f^{-1}(\mathbf{B}_-(D))$.
- (6) If $\{A_i\}$ is a sequence of ample divisors converging to 0 in $N^1(Y)$, with each $D + A_i$ a \mathbb{Q} -divisor, then $\mathbf{B}_-(D) = \bigcup_j \mathbf{B}_-(D + A_j)$.
- (7) If $D \in \overline{\text{Mov}}(X)$, then every component of $\mathbf{B}_-(D)$ has codimension at least 2.

3. NEFNESS IN FAMILIES OF \mathbb{R} -DIVISORS

In this section, we adopt the notation of Section 2, with $n = 2$ and $k = 10$. The proof of Theorem 1.2 is contained in Lemmas 3.1, 3.3, and 3.4.

The Cremona transformation $\text{Cr} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ has a resolution

$$\begin{array}{ccc} X & \xlongequal{\quad} & X' \\ \pi \downarrow & & \downarrow \pi' \\ \mathbb{P}^2 & \xrightarrow{\text{Cr}} & \mathbb{P}^2 \end{array}$$

Here both $\pi : X \rightarrow \mathbb{P}^2$ and $\pi' : X' \rightarrow \mathbb{P}^2$ are the blow-up of \mathbb{P}^2 at three points; we employ two notations to emphasize that the standard bases $\{h, e_1, e_2, e_3\}$ and $\{h', e'_1, e'_2, e'_3\}$ are different.

If C is any curve on X , then its class on X' in the new basis is given by $M([C])$ where

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

Let $\rho : \Sigma \dashrightarrow \Sigma$ be the map given by $(p_1, \dots, p_{10}) \mapsto (p_8, p_9, p_{10}, \text{Cr}(p_1), \dots, \text{Cr}(p_7))$, where $\text{Cr} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is the Cremona transformation centered at p_8, p_9 , and p_{10} . This map is regular off of the set $L \subset \Sigma$, the locus of \mathbf{p} with p_8, p_9 , and p_{10} collinear. Let \mathbf{p} be any point of $\Sigma \setminus L$, and set $\mathbf{q} = \rho(\mathbf{p})$. Write Π_σ for the permutation matrix for $\sigma = (8, 9, 10, 1, 2, 3, 4, 5, 6, 7)$, and consider the map $M_\sigma^{\mathbf{p}\mathbf{q}} : N^1(X_{\mathbf{p}}) \rightarrow N^1(X_{\mathbf{q}})$ given in the standard bases by

$$M_\sigma^{\mathbf{p}\mathbf{q}} = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & I_7 \end{array} \right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Pi_\sigma \end{array} \right),$$

where both of these are 11×11 block matrices, but with different block sizes. If C is a curve on $X_{\mathbf{p}}$, there is a curve on $X_{\mathbf{q}}$ lying in the class $M_\sigma^{\mathbf{p}\mathbf{q}}([C])$, obtained by cyclically reordering the points so the last three come first, and then taking the strict transform of C under a Cremona transformation centered at these points. If \mathbf{p} is in very general position, then the map $\Phi_{\mathbf{p}\mathbf{q}}^{-1} : N^1(X_{\mathbf{q}}) \rightarrow N^1(X_{\mathbf{p}})$ is an isomorphism which identifies the nef cones, and so $M_\sigma = \Phi_{\mathbf{p}\mathbf{q}}^{-1} \circ M_\sigma^{\mathbf{p}\mathbf{q}} : N^1(X_{\mathbf{p}}) \rightarrow N^1(X_{\mathbf{p}})$ satisfies $M_\sigma(\text{Nef}(X_{\mathbf{p}})) = \text{Nef}(X_{\mathbf{p}})$. Note too that M_σ preserves the intersection form on $N^1(X_{\mathbf{p}})$.

Lemma 3.1. *The map M_σ has characteristic polynomial $(t-1)t^5q(t+t^{-1})$, where $q(t) = t^5 - t^4 - 6t^3 + 5t^2 + 8t - 5$. M_σ has a unique eigenvalue $\lambda \approx 1.431$ of magnitude greater than 1. When the λ -eigenvector $C_{\lambda,\mathbf{p}}$ is written as $h - \sum_{i=1}^{10} r_i e_i$, the first three coefficients satisfy $r_1 + r_2 + r_3 > 1$.*

The divisor $C_{\lambda,\mathbf{p}}$ is nef on $X_{\mathbf{p}}$ for very general \mathbf{p} .

Proof. The inequality on the coefficients can be checked by computing an approximation of the eigenvalue and then expressing each of the coefficients as a rational function of λ . The claimed nefness then follows from Lemma 2.1. \square

Remark 3.2. In the standard coordinates, the divisor is approximately

$$C_\lambda \approx (1, -0.451, -0.440, -0.408, -0.315, -0.307, -0.285, -0.220, -0.215, -0.199, -0.154).$$

Let C_λ be the corresponding divisor $h - \sum_{i=1}^{10} r_i e_i$ on the total space \mathcal{X} . Though $C_{\lambda,\mathbf{p}}$ is nef for very general \mathbf{p} , we will see that if \mathbf{p} lies on any of countably many subvarieties V_n of Σ for which $X_{\mathbf{p}}$ contains (-2) -curves of certain classes, $C_{\lambda,\mathbf{p}}$ is not nef. Define V_0 to be the set of $\mathbf{p} \in \Sigma$ for which p_1, p_2 , and p_3 are collinear. If $\mathbf{p}_0 \in V_0$, there is a curve $\bar{\ell} \subset X_{\mathbf{p}_0}$ of class $C_0 = h - e_1 - e_2 - e_3$. Then $C_{\lambda,\mathbf{p}_0} \cdot \bar{\ell} = 1 - r_1 - r_2 - r_3 < 0$, and C_{λ,\mathbf{p}_0} is not nef. Similarly, $\mathbf{p}_1 = \rho(\mathbf{p}_0)$ is a configuration of points with the first six lying on a conic, and the strict transform of that conic on $X_{\mathbf{p}_1}$ has negative intersection with C_{λ,\mathbf{p}_1} . Generally, for $n \geq 0$ define $V_{n+1} \subset \Sigma$ to be the strict transform of V_n under ρ .

Lemma 3.3. *Each V_n is a prime divisor not equal to L , and V_m and V_n are distinct if $m \neq n$. For any point $\mathbf{p}_n \in V_n$, there exists a curve $\bar{\ell} \subset X_{\mathbf{p}_n}$ in the class $M_\sigma^n(C_0)$.*

Proof. To prove these sets are distinct, we will construct a sequence of points $\mathbf{p}_n \in V_n \setminus L$ such that $X_{\mathbf{p}_n}$ contains a curve lying in the class $M_\sigma^n(C_0)$, which is the unique rational curve

of self-intersection less than or equal -2 . Let $E \subset \mathbb{P}^2$ be a smooth elliptic curve. Construct $\mathbf{p}_0 \in V_0$ by choosing points on E such that p_1, p_2 , and p_3 are the points of intersection of E with some line ℓ meeting E transversely, and p_4, \dots, p_{10} have the property that if $3d - \sum_{i=1}^{10} m_i = 0$, the class $d\ell|_E - \sum_{i=1}^{10} m_i p_i$ is not linearly equivalent to 0 on E unless $m_4 = \dots = m_{10} = 0$. This condition will be met if these points are chosen to be very general. Write $\bar{\ell}$ and \bar{E} for the strict transforms of ℓ and E on $X_{\mathbf{p}_0}$.

Suppose that $C \sim d\pi^*h - \sum_{i=1}^{10} m_i e_i$ is a rational curve with $K_{X_{\mathbf{p}_0}} \cdot C \geq 0$. Since $K_{X_{\mathbf{p}_0}} \sim -\bar{E}$, we have $\bar{E} \cdot C \leq 0$, and so $\bar{E} \cdot C = 0$. Then $3d - \sum_{i=1}^{10} m_i = 0$, and the hypothesis on the points implies that $C \sim h - e_1 - e_2 - e_3$ is the curve $\bar{\ell}$. It follows that that under any sequence of Cremona transformations, the last three points will not become collinear; indeed, the strict transform of a line containing these three points would be a (-2) -curve on $X_{\mathbf{p}_0}$, but $\bar{\ell}$ is the only such. We may therefore define a sequence of points $\mathbf{p}_{n+1} \in V_{n+1}$ by taking $\mathbf{p}_{n+1} = \rho(\mathbf{p}_n)$. For each n , $\bar{\ell}$ is the unique (-2) -curve on $X_{\mathbf{p}_n}$, and lies in the class $M_\sigma^n(C_0)$, where $C_0 = h - e_1 - e_2 - e_3$. This implies that the divisors V_m are distinct.

A general point $\mathbf{p}_n \in V_n$ is of the form $\rho^n(\mathbf{p}_0)$ for some point $\mathbf{p}_0 \in V_0$. The strict transform on $X_{\mathbf{p}_0}$ of a line through the first three points of \mathbf{p}_0 has class C_0 , and this curve has class $M_\sigma^n(C_0)$ on $X_{\mathbf{p}_n}$. \square

Lemma 3.4. *If $\mathbf{p} \in V_n$, then $C_{\lambda, \mathbf{p}}$ is not nef.*

Proof. For any point $\mathbf{p} \in V_n$, there is a curve $C_n \subset X_{\mathbf{p}}$ with class $M_\sigma^n(C_0)$. Then

$$C_{\lambda, \mathbf{p}} \cdot C_n = \left(\frac{1}{\lambda^n} M_\sigma^n([C_{\lambda, \mathbf{p}}]) \right) \cdot M_\sigma^n(C_0) = \frac{1}{\lambda^n} C_{\lambda, \mathbf{p}} \cdot C_0 < 0. \quad \square$$

4. THE STANDARD CREMONA TRANSFORMATION AND ITS ITERATES

We now turn to the second example, and will employ the notation of Section 2 for blow-ups of \mathbb{P}^3 . Some notation from Section 3 will be reused in the new context. The example of Theorem 1.1 will be constructed as an eigenvector of a map $N^1(X) \rightarrow N^1(X)$ induced on a blow-up of \mathbb{P}^3 by a certain sequence of Cremona transformations.

The standard Cremona transformation of \mathbb{P}^3 centered at four non-coplanar points p_1, \dots, p_4 is the birational map $\text{Cr} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by

$$[X_1; X_2; X_3; X_4] \mapsto [X_1^{-1}; X_2^{-1}; X_3^{-1}; X_4^{-1}],$$

where the coordinates are chosen so the points p_i lie at the intersections of the coordinate hyperplanes. The map Cr is toric and is easily seen to have a resolution

$$\begin{array}{ccc} & Y & \\ p \swarrow & & \searrow p' \\ X & \xrightarrow{\quad \bar{\text{Cr}} \quad} & X' \\ \pi \downarrow & & \downarrow \pi' \\ \mathbb{P}^3 & \xrightarrow{\quad \text{Cr} \quad} & \mathbb{P}^3 \end{array}$$

Here both $\pi : X \rightarrow \mathbb{P}^3$ and $\pi' : X' \rightarrow \mathbb{P}^3$ are the blow-up of \mathbb{P}^3 at p_1, \dots, p_4 , with exceptional divisors E_i and E'_i respectively. Let F_i and F'_i denote the strict transforms on X and X' of planes through the three points other than p_i , and H and H' the pullbacks of $\mathcal{O}_{\mathbb{P}^3}(1)$. Take

l_{ij} and l'_{ij} to be the lines in \mathbb{P}^3 through p_i and p_j , and \bar{l}_{ij} and \bar{l}'_{ij} their strict transforms. The \bar{l}_{ij} are smooth rational curves with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\overline{\text{Cr}}$ is the flop of these six curves. More precisely, p is the blow-up of X along the six curves \bar{l}_{ij} , with exceptional divisors isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and these are contracted along the other ruling by p' . The strict transform of F_i under $\overline{\text{Cr}}$ is the exceptional divisor E'_i , while the strict transform of E_i is F'_i .

The indeterminacy locus of $\overline{\text{Cr}} : X \dashrightarrow X'$ is the union of the six curves \bar{l}_{ij} ; since this map is an isomorphism in codimension 1, taking strict transforms of divisors induces an isomorphism $M : N^1(X) \rightarrow N^1(X')$, as well as an isomorphism $N : N_1(X) \rightarrow N_1(X')$ defined by requiring $D \cdot C = MD \cdot NC$. This action has been studied by Laface and Ugaglia in connection with special linear systems of divisors on \mathbb{P}^3 [7],[8]. In that context M describes the change in the multiplicity of a divisor at prescribed points under Cremona transformations centered at those points.

Lemma 4.1. *The isomorphisms $M : N^1(X) \rightarrow N^1(X')$ and $N : N_1(X) \rightarrow N_1(X')$ are given in the standard bases $[H], [E_1], \dots, [E_4]$ for $N^1(X)$ and $N^1(X')$ by the matrices*

$$M = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ -2 & 0 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 3 & 2 & 2 & 2 & 2 \\ -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

Proof. For every i , we may write $H \sim F_i + \sum_{j \neq i} E_j$, and so $4H \sim \sum_{i=1}^4 F_i + 3 \sum_{j=1}^4 E_j$. Similarly, $4H' \sim \sum_{i=1}^4 F'_i + 3 \sum_{j=1}^4 E'_j$. Taking strict transforms yields $4M(H) = \sum_{i=1}^4 E'_i + 3 \sum_{j=1}^4 F'_j$, and $4M(H) - 12H' = -8 \sum_{j=1}^4 E'_j$. This gives $M(H) = 3H' - 2 \sum_{j=1}^4 E'_j$, which is the first column of M . For the other columns, write $M(E_i) = F'_i = H' - \sum_{j \neq i} E'_j$. The matrix for N is then determined by $M^t I_{1,4} N = I_{1,4}$. \square

If \mathbf{p} is a set of $k \geq 4$ points in linear general position (i.e. with no more than three coplanar), a Cremona transformation centered at the first four induces a small birational map which we again denote by $\overline{\text{Cr}} : X_{\mathbf{p}} \dashrightarrow X_{\mathbf{q}}$, where $\mathbf{q} = (p_1, \dots, p_4, \text{Cr}(p_5), \dots, \text{Cr}(p_k))$.

Corollary 4.2 ([7], Proposition 2.1, Corollary 3.9). *Suppose that \mathbf{p} is a k -tuple in \mathbb{P}^3 with no four points coplanar and consider the map $\overline{\text{Cr}} : X_{\mathbf{p}} \dashrightarrow X_{\mathbf{q}}$ induced by a standard Cremona transformation centered at the first four points.*

(1) *If D is any divisor on $X_{\mathbf{p}}$, then*

$$[\overline{\text{Cr}}(D)] = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & I_{k-4} \end{array} \right) ([D]),$$

where $\overline{\text{Cr}}(D)$ denotes the strict transform of D .

(2) *If C is any curve on $X_{\mathbf{p}}$ which does not meet the curves \bar{l}_{ij} which make up the indeterminacy locus of $\overline{\text{Cr}}$, then*

$$[\overline{\text{Cr}}(C)] = \left(\begin{array}{c|c} N & 0 \\ \hline 0 & I_{k-4} \end{array} \right) ([C]),$$

where $\overline{\text{Cr}}(C)$ denotes the strict transform of C .

Proof. The strict transform of E_i is E'_i for $i > 4$, so the coefficients on these divisors are unaffected, and (1) is just Lemma 4.1. (2) follows from the fact that if C is disjoint from the indeterminacy locus of $\overline{\text{Cr}}$, the intersection of C with a divisor is unchanged under strict transform, and $\begin{pmatrix} N & 0 \\ 0 & I_{k-4} \end{pmatrix}$ is the linear map which preserves the intersection form. \square

We now focus on the case that $k = 9$ points are blown up. The blow-up of \mathbb{P}^3 at $k \leq 7$ very general points is a Mori dream space, while the blow-up at $k \geq 8$ is not [10]. S. Cacciola et al. give generators of the movable cone for $k \leq 6$ [4], while A. Prendergast-Smith has studied the movable cone for $k = 8$ non-general points in connection with the Kawamata-Morrison cone conjecture [13].

If I is a 4-tuple from among the nine points, there is a birational map $\text{Cr}_I : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined as a standard Cremona transformation centered at the four points of I , inducing a small rational map $\overline{\text{Cr}}_I : X_{\mathbf{p}} \rightarrow X_{\mathbf{q}}$, where \mathbf{q} is obtained from \mathbf{p} by fixing the four points of I and transforming the others by Cr_I . Given a sequence $\mathbf{I} = (I_1, \dots, I_n)$ of 4-tuples from among the nine points, the composition $\text{Cr}_{\mathbf{I}} = \text{Cr}_{I_n} \circ \dots \circ \text{Cr}_{I_1}$ is not defined in general; four of the points might become coplanar under some $\text{Cr}_{I_{j-1}}$. However, if $\mathbf{p} = \mathbf{p}_0$ is in very general position, arbitrary compositions of Cremona transformations are defined. When the composition is defined, we write $\overline{\text{Cr}}_{I_j} : X_{\mathbf{p}_{j-1}} \dashrightarrow X_{\mathbf{p}_j}$ for the induced small rational maps of the blow-ups, and $\overline{\text{Cr}}_{\mathbf{I}} : X_{\mathbf{p}_0} \dashrightarrow X_{\mathbf{p}_n}$ for their composition.

If $\bar{\ell} \subset X_{\mathbf{p}}$ is the strict transform of a line through p_1 and p_2 , the numerical class of its strict transform under $\overline{\text{Cr}}_{\mathbf{I}}$ could be computed using Lemma 4.2 if it were known that the strict transform of $\bar{\ell}$ under $\overline{\text{Cr}}_{I_{j-1}} \circ \dots \circ \overline{\text{Cr}}_{I_1}$ is disjoint from the indeterminacy locus of $\overline{\text{Cr}}_{I_k}$ for every $k \leq j-1$. Laface and Ugaglia have shown that this is indeed the case for very general blow-ups, and for convenience we provide their proof.

Lemma 4.3 ([8], Lemma 2.6). *There exists a configuration \mathbf{p} such that the unique quadric Q_0 through p_1, \dots, p_9 is smooth, p_1 and p_2 lie on a ruling ℓ of Q_0 , and $\bar{\ell}$ is the unique rational curve on \bar{Q}_0 with $(C \cdot C)_{\bar{Q}_0} \leq -2$.*

Proof. Fix a smooth quadric $Q_0 \subset \mathbb{P}^3$, and choose p_1 and p_2 on a ruling ℓ . Let B_0 be a smooth elliptic curve of type $(2, 2)$ on Q_0 , and pick p_3, \dots, p_9 such that if $2(b_1 + b_2) - \sum_{i=1}^9 a_i = 0$, then $(b_1 r_1 + b_2 r_2)|_{B_0} - \sum_{i=1}^9 a_i p_i$ is not linearly equivalent to 0 on B_0 unless $a_3 = \dots = a_9 = 0$; this condition is met if these points are very general on B_0 . Suppose now that $C \sim b_1 \psi_0^* r_1 + b_2 \psi_0^* r_2 - \sum_{i=1}^9 a_i e_i$ is a curve with $(C \cdot C)_{\bar{Q}_0} \leq -2$. Since $-K_{\bar{Q}_0} \sim \bar{B}_0$ is effective, it must be that $K_{\bar{Q}_0} \cdot C = 0$, so $-2(b_1 + b_2) + \sum_{i=1}^9 a_i = 0$ and $C|_{\bar{B}_0} \sim (b_1 \psi_0^* r_1 + b_2 \psi_0^* r_2)|_{\bar{B}_0} - \sum_{i=1}^9 a_i p_i \sim 0$. The choice of points then guarantees that $a_3 = \dots = a_9 = 0$, which in turn implies that $C = \bar{\ell}_0$. \square

Theorem 4.4 ([8], Proposition 2.7). *Let $\mathbf{I} = (I_1, \dots, I_n)$ be a finite sequence of 4-tuples, and let ℓ be the line in \mathbb{P}^3 between p_1 and p_2 , with $\bar{\ell}$ its strict transform on $X = X_{\mathbf{p}}$. There exists an open subset $U_{\mathbf{I}} \subset \Sigma$ such that if \mathbf{p} is contained in $U_{\mathbf{I}}$, the following hold:*

- (1) *The composition $\text{Cr}_{\mathbf{I}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is well-defined.*
- (2) *If $\bar{\ell}$ is not contained in the indeterminacy locus of $\overline{\text{Cr}}_{\mathbf{I}}$, then for each $1 \leq j \leq n$, the strict transform $\bar{\ell}_{j-1} \subset X_{\mathbf{p}_{j-1}}$ is disjoint from the indeterminacy locus of $\overline{\text{Cr}}_{I_j}$.*

Proof. The set of points in Σ for which both conditions are satisfied is open, so it suffices to exhibit a single configuration of such points. Let $\mathbf{p} = \mathbf{p}_0$ be a configuration satisfying the conclusions of Lemma 4.3, so that in particular ℓ is a ruling of Q_0 . The following will be proved by induction on j , starting with $j = 1$:

- (i) The map $\overline{\text{Cr}}_{I_j} : X_{\mathbf{p}_{j-1}} \dashrightarrow X_{\mathbf{p}_j}$ is defined.
- (ii) $\bar{\ell}_{j-1}$ is disjoint from the indeterminacy locus of $\overline{\text{Cr}}_{I_j}$.
- (iii) $\bar{\ell}_j$ is the unique rational curve on \bar{Q}_j with $(C \cdot C)_{\bar{Q}_j} \leq -2$.

For (i), the transformation Cr_{I_j} is not defined if the four points at which it is centered are contained in a plane $H \subset \mathbb{P}^3$. Then $H|_{Q_{j-1}}$ is a rational curve of type $(1, 1)$ through these four points, the strict transform of which on \bar{Q}_{j-1} then has self-intersection less than or equal to -2 , which is impossible by induction. Now, by the choice of points, $\bar{\ell}_{j-1}$ is contained in the conic \bar{Q}_{j-1} . No two p_a and p_b can lie on a ruling r of \bar{Q}_{j-1} , for the strict transform would satisfy $(\bar{r} \cdot \bar{r})_{\bar{Q}_{j-1}} \leq -2$. Thus the line r between p_a and p_b meets Q transversely, and \bar{r} does not meet \bar{Q}_{j-1} or $\bar{\ell}_{j-1}$, as claimed by (ii). As the indeterminacy locus comprises strict transforms of lines between two points, we see that $\overline{\text{Cr}}_{I_j}|_{\bar{Q}_{j-1}} : \bar{Q}_{j-1} \rightarrow \bar{Q}_j$ is an isomorphism, and so $\bar{\ell}_j$ is again the only rational curve of self-intersection less than or equal -2 on \bar{Q}_j , proving (iii). \square

Remark 4.5. This composition of Cremona transformations may be simultaneously defined over the set $U_{\mathbf{I}} \subset \Sigma$. There is an induced map $\rho_{\mathbf{I}} : \mathcal{X} \dashrightarrow \mathcal{X}$ which acts on each fiber by $\overline{\text{Cr}}_{\mathbf{I}}$, and sends the fiber over \mathbf{p}_0 to that over \mathbf{p}_n .

We now consider compositions of Cremona transformations centered at judiciously chosen sequences of quadruples from the among nine points. Let $\sigma \in S_9$ be the permutation $(6, 7, 8, 9, 1, 2, 3, 4, 5)$, and take $I_j = (\sigma^{-j}(1), \dots, \sigma^{-j}(4))$. The composition $\text{Cr}_{I_j} \circ \dots \circ \text{Cr}_{I_1}$ could equivalently be realized by repeatedly making a Cremona transformation centered at p_6, \dots, p_9 and then cyclically permuting the indices so these points become p_1, \dots, p_4 .

Let $X = X_{\mathbf{p}}$ be the blow-up at a very general configuration \mathbf{p} . Define $M_{\sigma} : N^1(X) \rightarrow N^1(X)$ and $N_{\sigma} : N_1(X) \rightarrow N_1(X)$ by

$$M_{\sigma} = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & I_5 \end{array} \right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Pi_{\sigma} \end{array} \right), \quad N_{\sigma} = \left(\begin{array}{c|c} N & 0 \\ \hline 0 & I_5 \end{array} \right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Pi_{\sigma} \end{array} \right),$$

where Π_{σ} is the permutation matrix for σ . The class of the strict transform of a divisor D under $\overline{\text{Cr}}_{I_n} \circ \dots \circ \overline{\text{Cr}}_{I_1}$ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \Pi_{\sigma} \end{smallmatrix} \right)^{-n} M_{\sigma}^n([D])$. This strict transform is a divisor on a different blow-up $X_{\mathbf{q}}$ (as in Section 3), but if \mathbf{p} is very general then by Lemma 2.2 this defines an effective, movable class on X as well, and so $M_{\sigma}(\overline{\text{Mov}}(X)) = \overline{\text{Mov}}(X)$. Thus $M_{\sigma} : N^1(X) \rightarrow N^1(X)$ is a linear map which preserves the effective and movable cones. Similarly, if C is a curve with strict transforms disjoint from the indeterminacy loci, its strict transform has class $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \Pi_{\sigma} \end{smallmatrix} \right)^{-n} N_{\sigma}^n([C])$ by Corollary 4.2. The following lemma summarizes the essential properties of M_{σ} .

Lemma 4.6. *The linear transformation M_{σ} has characteristic polynomial $p(t) = (t+1)(t-1)t^4q(t+t^{-1})$, where $q(t) = t^4 - 3t^3 + 4t - 1$. M_{σ} has four real eigenvalues: 1 , -1 , $\lambda \approx 1.800$ and $1/\lambda$. When the λ -eigenvector D_{λ} is written as $H - \sum r_i E_i$, the first two coefficients satisfy $r_1 + r_2 > 1$.*

The 1-eigenspace is spanned by the canonical class K_X , and the only effective classes in this space are multiples of the strict transform \bar{Q} of the unique quadric through the nine points.

Proof. The claims about the eigenvalues are easily verified from the characteristic polynomial. To obtain the claimed inequality on the coefficients, one may express the components of the eigenvector in terms of the dominant eigenvalue and compute their approximate values.

Suppose that there exists a divisor $D \in |n(2H - \sum_{i=1}^9 E_i)|$, not equal to $n\bar{Q}$. Then $D|_{\bar{Q}}$ is a curve on \bar{Q} linearly equivalent to $2(\psi_0^* r_1 + \psi_0^* r_2) - 2n \sum_{i=1}^9 e_i$. The blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at nine points is isomorphic to the blow-up of \mathbb{P}^2 at ten, and under this identification this is the class of a curve of degree $3n$ with multiplicity n at each of ten points. However, there are no such curves for points in very general position. \square

Remark 4.7. To three decimal places, D_λ is given in components by

$$D_\lambda \approx (1, -0.640, -0.634, -0.615, -0.554, -0.355, -0.352, -0.341, -0.307, -0.197).$$

In fact, the eigenvalues are defined in a threefold tower of quadratic extensions of \mathbb{Q} and have explicit expressions in radicals.

5. THE GEOMETRY OF D_λ

Lemma 5.1. *The class D_λ lies in $\overline{\text{Mov}}(X)$ and spans an extremal ray on $\overline{\text{Eff}}(X)$.*

Proof. The two claims follow from Lemma 2.1 by taking $V = N^1(X)$ and $T = M_\sigma$, with $G = \overline{\text{Mov}}(X)$ and $G = \overline{\text{Eff}}(X)$ respectively. The hypothesis on the dominant eigenvalue is verified in Lemma 4.6. \square

Lemma 5.2 (= Theorem 1.1, (1), except Zariski denseness, (2)). *If \mathbf{p} is very general, there is an infinite set of curves $C_n \subset X = X_{\mathbf{p}}$ such that $D_\lambda \cdot C_n < 0$, and $\mathbf{B}_-(D_\lambda)$ is not closed.*

Proof. The strategy is to construct curves C_n in the classes $N_\sigma^n([C_0])$, where C_0 is a line through p_1 and p_2 . By Theorem 4.4, we can find a sequence of configurations \mathbf{p}_j , defined for all integers j , with $\mathbf{p}_0 = \mathbf{p}$ and such that the maps $\overline{\text{Cr}}_{I_j} : X_{\mathbf{p}_{j-1}} \dashrightarrow X_{\mathbf{p}_j}$ are defined for all j . We may additionally assume that if $\ell \subset X_{\mathbf{p}_j}$ is a line not contained in the indeterminacy locus of $\overline{\text{Cr}}_{I_{j+1}}$, then for all $k \geq 0$ the strict transform of ℓ on $X_{\mathbf{p}_{j+k}}$ is disjoint from the indeterminacy locus of $\overline{\text{Cr}}_{I_{j+k+1}}$.

Suppose that $\bar{\ell} \subset X_{\mathbf{p}_{-n}}$ is the strict transform of a line between p_i and p_j . By Theorem 4.4, as long as p_i and p_j are not among the base points of Cr_{I_1} , the composition $\overline{\text{Cr}}_{I_n} \circ \dots \circ \overline{\text{Cr}}_{I_1}$ is well-defined for all n , and the strict transforms of $\bar{\ell}$ are disjoint from the indeterminacy loci of the maps $\overline{\text{Cr}}_{I_j}$. Taking $\bar{\ell}$ to be the line between $p_{\sigma^n(1)}$ and $p_{\sigma^n(2)}$ on $X_{\mathbf{p}_{-n}}$, we thus obtain a curve $C_n \subset X$ of class $N_\sigma^n([C_0])$, where $[C_0] = h - e_1 - e_2$ is the class of a line through the first two points. Note that $\text{Cr}_{I_{-n+1}} : X_{\mathbf{p}_{-n}} \dashrightarrow X_{\mathbf{p}_{-n+1}}$ is centered at $p_{\sigma^{n-1}(1)}, \dots, p_{\sigma^{n-1}(4)}$. Since $\sigma^n(1) = \sigma^{n-1}(5)$ and $\sigma^n(2) = \sigma^{n-1}(6)$, $\bar{\ell}$ is not among the curves in the indeterminacy locus of $\overline{\text{Cr}}_{I_{-n+1}}$.

The computation of D_λ in Lemma 4.6 gives $D_\lambda \cdot C_0 = 1 - (r_1 + r_2) < 0$, and so

$$D_\lambda \cdot C_n = (\lambda^{-n} M_\sigma^n D_\lambda) \cdot (N_\sigma^n C_0) = \lambda^{-n} D_\lambda \cdot C_0 < 0.$$

By (3) of Lemma 2.5, each curve C_n is contained in $\mathbf{B}_-(D_\lambda)$. However, D_λ is movable and so $\mathbf{B}_-(D_\lambda)$ contains no divisors by (7) of the same lemma. It follows that $\mathbf{B}_-(D_\lambda)$ is a countable union of curves. \square

The first few classes $[C_n] = \delta h - \sum_i \mu_i e_i$ are given below.

n	δ	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
0	1	1	1	0	0	0	0	0	0	0
1	3	1	1	1	1	1	1	0	0	0
2	7	3	2	2	2	1	1	1	1	1
3	13	4	4	4	4	3	2	2	2	1
4	25	8	8	8	7	4	4	4	4	3
5	45	14	14	14	13	8	8	8	7	4

Remark 5.3. Each of the curves C_n may be simultaneously constructed on $X_{\mathbf{p}}$ for all \mathbf{p} in an open set $U_n \subset \Sigma$ by Remark 4.5. Let $\mathcal{C}_{0,n}$ be the union of the lines through $p_{\sigma^n(1)}$ and $p_{\sigma^n(2)}$ in every fiber, and take \mathcal{C}_n its strict transform under the map $\rho_{\mathbf{I}}$. The fiber of \mathcal{C}_n over \mathbf{p} is the curve C_n constructed above, which we write as $C_{n,\mathbf{p}}$ to emphasize the base points.

Lemma 5.4 (= Theorem 1.1, (1), denseness). *The curves $C_{n,\mathbf{p}}$ are Zariski dense on $X_{\mathbf{p}}$.*

Proof. Suppose that the closure has positive codimension. Let $\overline{\text{Cr}}_{\mathbf{I}_9} = \overline{\text{Cr}}_{I_9} \circ \cdots \circ \overline{\text{Cr}}_{I_1} : X_{\mathbf{p}} \dashrightarrow X_{\mathbf{q}}$ be composition of the first nine Cremona transformations considered above. Take S to be the codimension 1 part of $\overline{\bigcup_{n \geq 0} C_{n,\mathbf{p}}}$ and T the codimension 1 part of $\overline{\bigcup_{n \geq 0} C_{n,\mathbf{q}}}$. If D is an effective divisor on $X_{\mathbf{p}}$, its strict transform under $\overline{\text{Cr}}_{\mathbf{I}_9}$ has class $\Pi_{\sigma}^{-9} M_{\sigma}^9([D]) = M_{\sigma}^9([D])$. From the construction of Lemma 5.2, we see that the strict transform of $C_{n,\mathbf{p}}$ under $\overline{\text{Cr}}_{\mathbf{I}_9}$ is $C_{n+9,\mathbf{q}}$, and so

$$\overline{\bigcup_{n \geq 0} C_{n,\mathbf{q}}} = \bigcup_{n=0}^8 C_{n,\mathbf{q}} \cup \overline{\text{Cr}}_{\mathbf{I}_9} \left(\overline{\bigcup_{n \geq 0} C_{n,\mathbf{p}}} \right).$$

In particular, T is the strict transform under $\overline{\text{Cr}}_{\mathbf{I}_9}$ of S , and so $[T] = (\Phi_{\mathbf{p}\mathbf{q}} \circ M_{\sigma}^9)([S])$.

By Lemma 2.3, there is a family $\bar{\mathcal{C}} \rightarrow \Sigma$ such that $\overline{\bigcup_{n \geq 0} C_{n,\mathbf{p}}} = \bar{\mathcal{C}}_{\mathbf{p}}$ for every \mathbf{p} off of some countable union of subvarieties, where $\bar{\mathcal{C}}_n$ is the closure in \mathcal{X} of the family \mathcal{C}_n constructed in Remark 4.5. As a result, S and T are both fibers over Σ of some subvariety of \mathcal{X} , and $\Phi_{\mathbf{p}\mathbf{q}}([S]) = [T]$. This implies that $[S]$ lies in the 1-eigenspace of M_{σ}^9 . By Lemma 4.6, the only effective \mathbb{Q} -divisors in this class are multiples of $\bar{Q} \sim -K_{X_{\mathbf{p}}}/2$, the strict transform of the unique quadric through nine points, and so it must be that $S = \bar{Q}$. However, as the points are in very general position, the line $C_{0,\mathbf{p}}$ through the first two is not contained in \bar{Q} , and so neither are any of the $C_{n,\mathbf{p}}$. This means that \bar{Q} is not a component of the Zariski closure. It follows that the Zariski closure has no components of codimension 1, which is impossible since it contains infinitely many curves. \square

6. ZARISKI DECOMPOSITION OF D_{λ}

The non-closedness of $\mathbf{B}_{-}(D_{\lambda})$ further implies that D_{λ} admits no Zariski decomposition in several standard senses. We recall the form of decomposition in dimension two:

Theorem 6.1 (Zariski decomposition theorem, e.g. [14]). *Let D be a pseudoeffective \mathbb{R} -divisor on a smooth projective surface X . There exists an effective divisor $N = \sum_i a_i N_i$ such that $P = D - N$ is nef, $(N_i \cdot N_j)$ is negative definite, and $P \cdot N_i = 0$.*

There are several analogues of Zariski decompositions for divisors on higher-dimensional varieties, imposing conditions which ensure the retention of useful properties of the two-dimensional version. One decomposition which always exists and has proved important is the divisorial Zariski decomposition of a pseudoeffective \mathbb{R} -divisor D , due to Nakayama.

Definition 6.2 ([12]). Suppose that D is an \mathbb{R} -divisor. For a prime divisor E on X , let

$$\sigma_E(D) = \sup_{A \text{ ample}} \left(\min_{D' \equiv_{\text{num}} D + A} \text{ord}_E(D') \right).$$

Set $N_\sigma(D) = \sum_E \sigma_E(D) \cdot E$, and $P_\sigma(D) = D - N_\sigma(D)$. This is a finite sum, and $P_\sigma(D) \in \overline{\text{Mov}}(X)$. When D is a big \mathbb{Q} -divisor, in fact $\sigma_E(D) = \min_{D' \equiv_{\text{num}} D} \text{ord}_E(D')$.

In dimension two, this coincides with the standard Zariski decomposition, but in higher dimensions $P_\sigma(D)$ is only movable and not in general nef. To obtain a closer analogue of the Zariski decomposition, given a pseudoeffective \mathbb{R} -divisor on a smooth variety X , one might ask for a birational modification $f : Y \rightarrow X$ and a decomposition $f^*D \equiv_{\text{num}} P + N$, with P is nef and N effective. This is termed a *weak Zariski decomposition* by Birkar. Additionally, we might replace numerical equivalence by linear equivalence and ask that:

- (1) CKM: the maps $H^0(Y, \mathcal{O}_Y(\lfloor mP \rfloor)) \rightarrow H^0(Y, \mathcal{O}_Y(\lfloor mf^*D \rfloor))$ are all isomorphisms.
- (2) Fujita: if $g : Y' \rightarrow Y$ is birational, and $P' \leq g^*f^*D$ is nef, then $P' \leq g^*P$.
- (3) Nakayama: $P = P_\sigma(f^*D)$ is the positive part of the divisorial Zariski decomposition.

The survey [14] of Prokhorov introduces the important properties of these and other higher-dimensional versions of the Zariski decomposition. Nakayama constructed an example of an \mathbb{R} -divisor on a \mathbb{P}^2 -bundle over an abelian surface which admits no Zariski decomposition any of these three senses [12]. However, the divisor of Nakayama's example is itself big, thus effective, and trivially admits a weak Zariski decomposition. We will show that D_λ does not admit a weak Zariski decomposition, and give a related example of a big \mathbb{R} -divisor on a smooth fourfold which does not admit a Zariski decomposition in any of the three above senses.

Birkar has shown that the existence of even a weak Zariski decomposition has implications for the minimal model program: if (X, B) is a n -dimensional lc pair such that $K_X + B$ admits a weak Zariski decomposition, then (X, B) has a terminal model, assuming the LMMP in dimension $n - 1$ [2]. We observe that the divisors admitting such a decomposition constitute a cone in $N^1(X)$.

Definition 6.3. The b-Nef cone is the union

$$\text{b-Nef}(X) = \bigcup_{\substack{f: Y \rightarrow X \\ \text{birational}}} f_*(\text{Nef}(Y)).$$

Lemma 6.4. *The set $\text{b-Nef}(X)$ is a (not necessarily closed) convex cone, with $\text{int}(\overline{\text{Mov}}(X)) \subseteq \text{b-Nef}(X) \subseteq \overline{\text{Mov}}(X)$. The set of divisors admitting a weak Zariski decomposition is the cone $\text{b-Nef}(X) + \overline{\text{Eff}}(X) \subseteq \overline{\text{Eff}}(X)$.*

Proof. That $\text{b-Nef}(X)$ is a cone follows from the fact that any two models admit a common resolution. If $f : Y \rightarrow X$ is a resolution of the base ideal of a Cartier divisor D with no fixed component, and $f^*D = P + F$ the decomposition into basepoint-free and fixed components, then P is nef with $f_*P = D$. Any \mathbb{Q} -divisor in the interior of $\overline{\text{Mov}}(X)$ has a multiple with

base locus of codimension 2 [6, Lemma 2.2], so this gives the first containment. The second follows from the fact that the pushforward of a nef divisor is movable.

Suppose that D admits a weak Zariski decomposition $f^*D = P + N$. Then $D = f_*P + N$ and $D \in \text{b-Nef}(X) + \text{Eff}(X)$. Conversely, suppose $D = f_*P + N$, so that $f^*D = P + E + f^*N$, with E exceptional. Then $-E$ is f -nef, and by the negativity lemma E is effective, so $f^*D = P + (E + f^*N)$ is a weak Zariski decomposition. \square

Remark 6.5. If $\text{b-Nef}(X)$ is closed, then $\text{b-Nef}(X) = \overline{\text{Mov}}(X)$, and every class in $\overline{\text{Mov}}(X)$ is the pushforward of a nef divisor. In this situation every pseudoeffective divisor admits a weak Zariski decomposition: the divisorial Zariski decomposition gives $D = P_\sigma(D) + N_\sigma(D) = f_*P + N_\sigma(D)$. This is the case, for example, when $\dim X = 2$ or X is a Mori dream space.

Lemma 6.6 (= Theorem 1.1, (3)). D_λ does not admit a weak Zariski decomposition.

Proof. The following conditions are sufficient to guarantee that a divisor D admits no weak Zariski decomposition.

- (1) $[D]$ spans an extremal ray on $\overline{\text{Eff}}(X)$.
- (2) D is not numerically equivalent to any effective divisor.
- (3) There does not exist birational $f : Y \rightarrow X$ and a nef \mathbb{R} -divisor P on Y such that $f_*P = D$.

The first of these is demonstrated in Lemma 5.1, and since D_λ is extremal and not a rational class, it is not effective. It remains only to check the last condition. Suppose that $D_\lambda = f_*P$; then $f^*D_\lambda = P + E$ where E is f -exceptional, and E is effective by the negativity lemma. For each n , pick a curve \tilde{C}_n on Y mapping finitely to C_n , and let $d_n = \deg(\tilde{C}_n \rightarrow C_n)$. Only finitely many of the \tilde{C}_n are contained in $\text{Supp } E$, but for any curve \tilde{C}_n not contained in $\text{Supp } E$, we have $\tilde{C}_n \cdot E \geq 0$, and so compute $d_n(D_\lambda \cdot C_n) = f^*D_\lambda \cdot \tilde{C}_n = P \cdot \tilde{C}_n + E \cdot \tilde{C}_n \geq 0$, a contradiction. \square

Though the divisor D_λ is not big, a standard construction gives a big \mathbb{R} -divisor on a smooth 4-fold with non-closed diminished base locus. Fix an embedding $X \rightarrow \mathbb{P}^N$, let $CX \subset \mathbb{P}^{N+1}$ be the projective cone over X , and take $p : Y \rightarrow CX$ the blow-up at the cone point. The map p is birational with a unique exceptional divisor $E \cong X$; write $i_E : X \rightarrow Y$ for the inclusion. The variety Y has the structure of a \mathbb{P}^1 -bundle $q : Y \cong \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(1)) \rightarrow X$.

Lemma 6.7. *There exists a big \mathbb{R} -divisor D'_λ on Y with $\mathbf{B}_-(D'_\lambda)$ a countable union of curves. Moreover, $D'_\lambda \subset Y$ does not admit a Zariski decomposition in the sense of CKM, Fujita, or Nakayama.*

Proof. Let H be an ample divisor on CX with support disjoint from the cone point, and set $D'_\lambda = p^*H + q^*D_\lambda$. Choosing H sufficiently large, we may assume the base locus of D'_λ is contained in E . Observe that D'_λ is the sum of a big divisor and a pseudoeffective one, and thus big. Properties (2), (4), and (5) of Theorem 2.5 imply that $\mathbf{B}_-(D'_\lambda) \subseteq \mathbf{B}_-(p^*H) \cup \mathbf{B}_-(q^*D_\lambda) = \mathbf{B}_-(q^*D_\lambda) \subseteq q^{-1}\mathbf{B}_-(D_\lambda)$. On the other hand, the choice of H implies that $\mathbf{B}_-(D'_\lambda) \subseteq E$.

Each curve $C'_j = i_E(C_j)$ has $C'_j \cdot D'_\lambda = q(C'_j) \cdot D_\lambda < 0$, and so $C'_j \subset \mathbf{B}_-(D'_\lambda)$. If $\mathbf{B}_-(D'_\lambda)$ is Zariski closed, it must contain the closure of the curves C'_j , which is the entire divisor E by Theorem 5.4. But E is not contained in $q^{-1}(\mathbf{B}_-(D_\lambda))$, so this is impossible, and it must be that $\mathbf{B}_-(D'_\lambda)$ is a countable union of curves in E .

For the second claim, it suffices to show there is no Zariski decomposition in the sense of Nakayama [12, 2.1.10(3)]. Suppose that $f : Y' \rightarrow Y$ is such that $P_\sigma(f^*D'_\lambda)$ is nef. Any $f^*(D'_\lambda)$ -negative curve is contained in $\text{Supp } N_\sigma(f^*D'_\lambda)$, but since D'_λ is movable, $N_\sigma(f^*D'_\lambda)$ is f -exceptional. Thus all the D'_λ -negative curves in Y must be contained in the image of the exceptional locus, an impossibility. \square

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